Discrimination with error margin between two states
- Case of general occurrence probabilities -

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We investigate a state discrimination problem which interpolates minimum-error and unambiguous discrimination by introducing a margin for the probability of error. We closely analyze discrimination of two pure states with general occurrence probabilities. The optimal measurements are classified into three types. One of the three types of measurement is optimal depending on parameters (occurrence probabilities and error margin). We determine the three domains in the parameter space and the optimal discrimination success probability in each domain in a fully analytic form. It is also shown that when the states to be discriminated are multipartite, the optimal success probability can be attained by local operations and classical communication. For discrimination of two mixed states, an upper bound of the optimal success probability is obtained.

I. INTRODUCTION

Distinguishing quantum states in various situations is a fundamental and highly nontrivial problem in quantum information theory. This is because quantum measurement is statistical in nature and it generally destroys the state of the system to be measured.

Quantum state discrimination [1] is one of such problems. In this problem, we are given an unknown quantum state ρ, which is chosen from a set of known states $\{\rho_a\}$ with some known occurrence probabilities. The task is to find the optimal measurement scheme to identify the given state ρ with one in the set $\{\rho_a\}$. Two settings have been commonly investigated. In minimum-error discrimination, the discrimination success probability is maximized without any constraint on the probability of erroneous results [2]. In unambiguous discrimination, however, the success probability is maximized under the condition that measurement should not produce erroneous results, which is possible by allowing an inconclusive result "I don't know" [3, 4, 5, 6]. Other interesting alternative approaches include a maximum-confidence measurement analyzed in Ref. [7] and the scheme considered in Refs. [8, 9, 10, 11], in which the probability of correct discrimination is maximized while the rate of inconclusive results is fixed.

We consider a problem of maximizing the success probability under the condition that the probability of error should not exceed a certain error margin m [12, 13]. It is clear that unambiguous discrimination is formulated as the case of $m = 0$, while minimum-error discrimination corresponds to the case of $m = 1$. By controlling the error margin, this scheme continuously interpolates the minimum-error and unambiguous discrimination problems. Touzel, Adamson, and Steinberg [12] compared the numerical results of projective and positive operator-valued measure (POVM) measurements in this scheme. In our previous paper [13], we analyzed discrimination with error margin between two pure states with equal occurrence probabilities and obtained the optimal success probability in a closed analytic form.

In this paper, we extend the analysis of our previous paper [13] to the case of general occurrence probabilities. A new feature is that the two-dimensional parameter space consisting of occurrence probabilities and the error margin is divided into three domains. The types of optimal measurement differ depending on the domain. Suppose the error margin is so large that the constraint on the probability of error is inactive. Then, the optimal measurement is expected to be that of minimum-error discrimination. Hereafter, the domain where this is the case is called minimum-error domain. To see what happens when the error margin is small, let us recall the results of unambiguous discrimination ($m = 0$). If the occurrence probability of one of the states is sufficiently small, the optimal measurement produces only two outcomes omitting this state. For general error margin, this is expected to happen in a domain of the parameter space, which we call single-state domain. Intermediate domain is the one where probabilities of three measurement outcomes are non zero.

The main purpose of this paper is to determine these three domains and the optimal success probability in each domain in a fully analytic form. The problem is formulated and the main results are presented in Sec. II. Derivation of the results is detailed in Secs. III and IV.

We can consider two types of error margin for the probability of error. One is the constraint on the mean probability of error, which will be discussed first. The other is the constraint on conditional error probabilities. In Sec. V, we establish a relation between the optimal success probabilities of the two types of constraint. We also discuss discrimination of two mixed states. In Sec. VI, we show that an upper bound of the success probability for two mixed states can easily be obtained in terms of the optimal success probability of two pure states.

II. PROBLEM AND SOLUTION

We consider the discrimination problem between two pure states $\rho_1 = |\phi_1\rangle\langle\phi_1|$ and $\rho_2 = |\phi_2\rangle\langle\phi_2|$ with oc-

currence probabilities η_1 and η_2, respectively. To avoid trivial exceptional cases, we assume that $\eta_1 \neq 0$ and $\eta_2 \neq 0$. We also assume that the two states are linearly independent and we work in the two-dimensional subspace V spanned by these two states. The measurement is described by a positive operator-valued measure on V, which consists of three elements $\{E_\mu\}_{\mu=1}^3$. Measurement outcome labeled by $\mu = 1$ or 2 means that the given input state is identified with state ρ_μ. Element E_3 produces the inconclusive result. Let us denote by P_{ρ_a, E_μ} the joint probability that the given state is ρ_a ($a = 1, 2$) and the measurement outcome is μ. The probability P_{ρ_a, E_μ} is given by

$$P_{\rho_a, E_\mu} = \eta_a \mathrm{tr} E_\mu \rho_a.$$

The discrimination success probability $p_\circ$ and the mean probability of error $p_\times$ are given by

$$\begin{aligned} p_\circ &\equiv P_{\rho_1, E_1} + P_{\rho_2, E_2} \\ &= \eta_1 \mathrm{tr} E_1 \rho_1 + \eta_2 \mathrm{tr} E_2 \rho_2, \quad (1) \\ p_\times &\equiv P_{\rho_1, E_2} + P_{\rho_2, E_1} \\ &= \eta_1 \mathrm{tr} E_2 \rho_1 + \eta_2 \mathrm{tr} E_1 \rho_2. \quad (2) \end{aligned}$$

We require that the mean probability of error $p_\times$ must not exceed a certain error margin m ($0 \leq m \leq 1$). Then, the task is to maximize the success probability $p_\circ$ under the conditions:

$$\begin{aligned} E_1 &\geq 0, \quad E_2 \geq 0, \quad E_3 \geq 0, \quad &(3a) \\ E_1 &+ E_2 + E_3 = 1, \quad &(3b) \\ p_\times &\leq m, \quad &(3c) \end{aligned}$$

where Eqs. (3a, 3b) are the usual conditions for a POVM.

This problem can be formulated as one of semidefinite programming (SDP). See Ref. [14] for a general review and Refs. [15, 16] for applications of SDP to quantum-state discrimination. According to the general theory of SDP, we can write the necessary and sufficient conditions for the optimal POVM. For our purpose, it suffices to see that they are sufficient conditions.

Suppose a Hermitian operator Y acting on V and a real number y satisfy conditions

$$\begin{aligned} Y &\geq 0, \quad &(4a) \\ Y &\geq \eta_1 \rho_1 - y \eta_2 \rho_2, \quad &(4b) \\ Y &\geq \eta_2 \rho_2 - y \eta_1 \rho_1, \quad &(4c) \\ y &\geq 0. \quad &(4d) \end{aligned}$$

It is easy to show that

$$d \equiv \mathrm{tr} Y + my, \quad (5)$$

gives an upper bound for the success probability $p_\circ$, because

$$\begin{aligned} p_\circ &= \eta_1 \mathrm{tr} E_1 \rho_1 + \eta_2 \mathrm{tr} E_2 \rho_2 \\ &\leq \mathrm{tr} E_1 (Y + y \eta_2 \rho_2) + \mathrm{tr} E_2 (Y + y \eta_1 \rho_1) \\ &= \mathrm{tr}(E_1 + E_2) Y + y p_\times \\ &\leq \mathrm{tr} Y + ym = d. \end{aligned}$$

It is clear that this upper bound is attained if and only if the following relations hold:

$$\begin{aligned} E_1(Y - (\eta_1 \rho_1 - y \eta_2 \rho_2)) &= 0, \quad &(6a) \\ E_2(Y - (\eta_2 \rho_2 - y \eta_1 \rho_1)) &= 0, \quad &(6b) \\ E_3 Y &= 0, \quad &(6c) \\ y(m - p_\times) &= 0. \quad &(6d) \end{aligned}$$

Thus, the set of equations given by Eqs. (3), (4), and (6) is a sufficient condition for an optimal solution. As we will see, we can construct a solution satisfying this condition for any parameters: η_a, m and $\langle \phi_1 | \phi_2 \rangle$. The general theory SDP shows it is also a necessary condition [14, 15, 16]. Minimizing d under conditions Eqs. (4) is called dual problem, whereas the original problem of maximizing $p_\circ$ under conditions Eqs. (3) is referred to as primal problem.

Let us begin by looking at ranks of optimal POVM elements, which are operators on the two-dimensional space V. We note that they are of rank 1 at most. This can be seen in the following way. Suppose that E_3 is of rank 2. Condition Eq. (6c) requires that $Y = 0$. Then, from Eqs. (4b, 4c), we find $y \eta_2 \rho_2 \geq \eta_1 \rho_1$ and $y \eta_1 \rho_1 \geq \eta_2 \rho_2$. It is easy to see that these inequalities contradict the assumption that the two states are linearly independent. Next, suppose that E_1 is of rank 2. From Eq. (6a), we have $Y = \eta_1 \rho_1 - y \eta_2 \rho_2$. Then, Eqs. (4a, 4c) require that $\eta_1 \rho_1 \geq y \eta_2 \rho_2$ and $\eta_1 \rho_1 \geq \eta_2 \rho_2$, which are again inconsistent with the linear independence of the two states and the assumption that $\eta_2 \neq 0$. It is clear that the rank of E_2 is also 1 at most.

As stated in Sec. I, there are three types of measurements, one of which becomes optimal depending on domains of the parameter space of occurrence probabilities and error margin. This classification can be done according to the ranks of POVM. In the minimum-error domain, the optimal POVM is that of minimum-error discrimination, which implies that ranks of E_1 and E_2 is 1 while $E_3 = 0$. In the single-state domain, optimal measurement produces only two outcomes omitting one of the two states. In this case, either E_1 or E_2 is 0 and the remaining two POVM elements are of rank 1. The intermediate domain is where all POVM elements are of rank 1 and probabilities of obtaining the three outcomes are non zero.

In what follows, we present the main results first, leaving their derivation to subsequent sections. We assume that $\eta_1 \leq \eta_2$ without loss of generality. To make expressions simpler, we define

$$\begin{aligned} S &\equiv |\langle \phi_1 | \phi_2 \rangle|^2, \quad &(7) \\ T &\equiv 1 - |\langle \phi_1 | \phi_2 \rangle|^2. \quad &(8) \end{aligned}$$

The parameter space is divided into the following three domains:

$$\begin{cases} \text{Minimum-error domain}: & m_c \leq m \leq 1, \\ \text{Intermediate domain}: & m'_c \leq m \leq m_c, \\ \text{Single-state domain}: & 0 \leq m \leq m'_c, \end{cases}$$
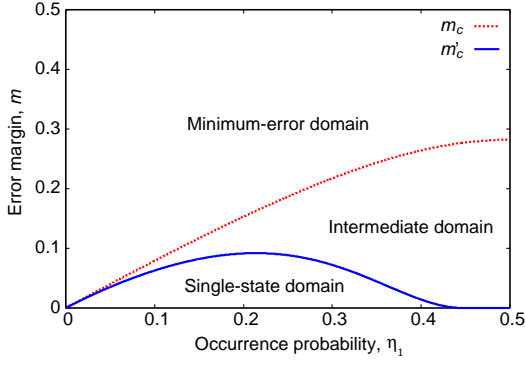
FIG. 1: The three domains in the parameter space of occurrence probability η_1 and error margin m: minimum-error domain, intermediate domain, and single-state domain. Fidelity $|\langle \phi_1 | \phi_2 \rangle|$ is taken to be 0.9.

where two critical error margins m_c and m_c' are defined by

$$m_c \equiv \frac{1}{2}\left(1 - \sqrt{1 - 4\eta_1\eta_2 S}\right), \tag{9}$$

$$m_c' \equiv \begin{cases} \dfrac{(\eta_1 - \sqrt{\eta_1\eta_2 S})^2}{1 - 2\sqrt{\eta_1\eta_2 S}} & (\eta_1 \leq \eta_2 S), \\ 0 & (\eta_1 \geq \eta_2 S). \end{cases} \tag{10}$$

Figure 1 depicts the three domains in the case of $\sqrt{S} = |\langle \phi_1 | \phi_2 \rangle| = 0.9$. The optimal discrimination success probability in each domain is found to be

$$p_{\max} = \begin{cases} \frac{1}{2}\left(1 + \sqrt{1 - 4\eta_1\eta_2 S}\right) & (m_c \leq m \leq 1), \\ \left(\sqrt{m} + \sqrt{1 - 2\sqrt{\eta_1\eta_2 S}}\right)^2 & (m_c' \leq m \leq m_c), \\ \eta_2\left(\sqrt{\frac{m}{\eta_1}S} + \sqrt{\frac{\eta_1 - m}{\eta_1}T}\right)^2 & (0 \leq m \leq m_c'). \end{cases} \tag{11}$$

The critical margin m_c is actually the mean error probability of optimal minimum-error discrimination. If $m_c \leq m$, the constraint on the probability of error is inactive. This is the reason why $p_{\max}$ in the minimum-error domain is given by that of minimum-error discrimination. In Fig. 2, we plot the optimal success probability $p_{\max}$ and $\mathrm{tr}E_1$ against error margin m for a fixed η_1. The plot of $\mathrm{tr}E_1$ clearly shows the border between the single-state and intermediate domains, though the curve of $p_{\max}$ is smooth at $m = m_c'$.

In unambiguous discrimination ($m = 0$), for a sufficiently small η_1, the optimal measurement is always of the single-state type. Intuitively, this appears reasonable. However, Fig. 1 shows that this is no longer true for a finite error margin. For example, fix m to be around 0.06 and vary η_1 from 0.5 to 0. Then, the type of optimal measurement varies in a nontrivial way: from the intermediate to single-state, intermediate, and minimum-error type.

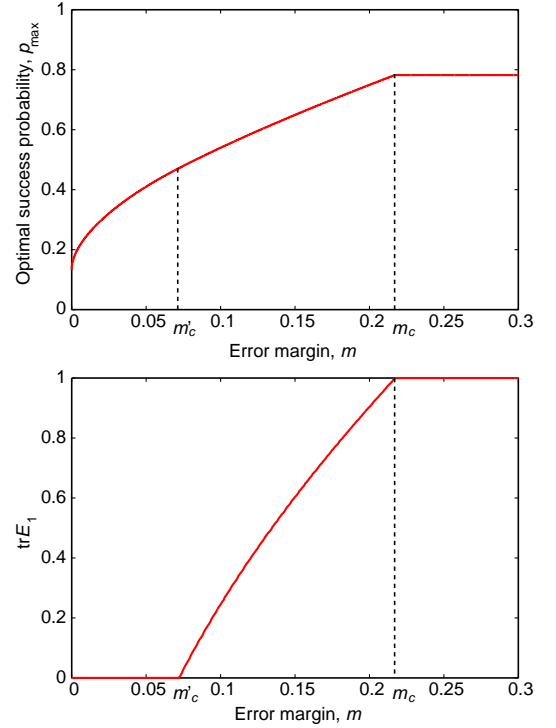Figure 3 displays a three-dimensional overview of the optimal success probability.

FIG. 2: The optimal success probability $p_{\max}$ (upper part) and $\mathrm{tr}E_1$ (lower part) vs error margin m. The occurrence probability η_1 is 0.3 and fidelity $|\langle \phi_1 | \phi_2 \rangle|$ is 0.9.
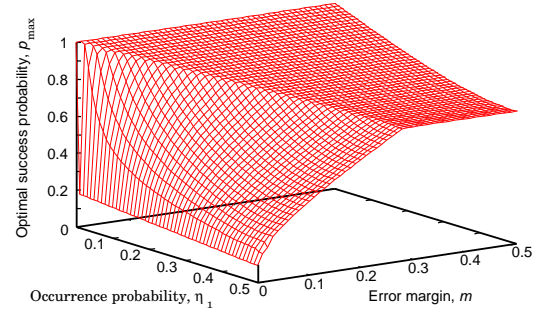

FIG. 3: Three-dimensional plot of the optimal success probability $p_{\max}$ vs occurrence probability η_1 and error margin m. Fidelity $|\langle \phi_1 | \phi_2 \rangle|$ is taken to be 0.9.

III. INTERMEDIATE DOMAIN

In this section, we construct a solution where all POVM elements E_1, E_2, and E_3 are non zero and of rank 1. The attainability conditions given by Eqs. (6a-6c) imply that positive semidefinite operators

$$Y_1 \equiv Y - (\eta_1\rho_1 - y\eta_2\rho_2),$$
$$Y_2 \equiv Y - (\eta_2\rho_2 - y\eta_1\rho_1),$$

and Y are all of rank 1. It is convenient to use the Bloch vector representation for ρ_a and other operators acting

on V.

$$\rho_a = \frac{1 + \boldsymbol{n}_a \cdot \boldsymbol{\sigma}}{2} \ (a = 1, 2),$$

where $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are Pauli's matrices. Writing

$$Y = \alpha + \boldsymbol{\beta} \cdot \boldsymbol{\sigma},$$

we have

$$Y_1 = \alpha - \frac{\eta_1 - y\eta_2}{2} + \left(\boldsymbol{\beta} - \frac{\boldsymbol{a}_1}{2}\right) \cdot \boldsymbol{\sigma},$$
$$Y_2 = \alpha - \frac{\eta_2 - y\eta_1}{2} + \left(\boldsymbol{\beta} - \frac{\boldsymbol{a}_2}{2}\right) \cdot \boldsymbol{\sigma},$$

where we introduced two vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ defined to be

$$\boldsymbol{a}_1 = \eta_1 \boldsymbol{n}_1 - y\eta_2 \boldsymbol{n}_2, \qquad (12)$$
$$\boldsymbol{a}_2 = \eta_2 \boldsymbol{n}_2 - y\eta_1 \boldsymbol{n}_1. \qquad (13)$$

Since the smaller eigenvalues of operators Y_1, Y_2, and Y are all zero, we obtain the following three equations for α and $\boldsymbol{\beta}$:

$$\alpha - \frac{\eta_1 - y\eta_2}{2} = \left|\boldsymbol{\beta} - \frac{\boldsymbol{a}_1}{2}\right|, \qquad (14\text{a})$$

$$\alpha - \frac{\eta_2 - y\eta_1}{2} = \left|\boldsymbol{\beta} - \frac{\boldsymbol{a}_2}{2}\right|, \qquad (14\text{b})$$

$$\alpha = |\boldsymbol{\beta}|. \qquad (14\text{c})$$

Solving Eqs. (14) requires a rather long calculation. It turns out that parameter y must satisfy $y \geq 1$ and vector $\boldsymbol{\beta}$ is given by

$$\boldsymbol{\beta} = \frac{y}{2(y-1)} \left\{ \left(\eta_1 \pm \sqrt{\frac{\eta_1 \eta_2}{S}}\right) \boldsymbol{n}_1 \right.$$
$$\left. + \left(\eta_2 \pm \sqrt{\frac{\eta_1 \eta_2}{S}}\right) \boldsymbol{n}_2 \right\}, \qquad (15)$$

and α is given by

$$\alpha = |\boldsymbol{\beta}| = \frac{y}{2(y-1)} \left(1 \pm 2\sqrt{\eta_1 \eta_2 S}\right). \qquad (16)$$

Now that we have Y and y satisfying Eqs. (4), we obtain an upper bound for the success probability by calculating $d = \mathrm{tr}Y + ym$.

$$d = \mathrm{tr}Y + ym = \frac{y}{y-1}\left(1 \pm 2\sqrt{\eta_1 \eta_2 S}\right) + my. \qquad (17)$$

We determine parameter y so that the upper bound d is minimized, which leads to

$$d = \left(\sqrt{m} + \sqrt{1 \pm 2\sqrt{\eta_1 \eta_2 S}}\right)^2, \qquad (18)$$

$$y = 1 + \frac{\sqrt{1 \pm 2\sqrt{\eta_1 \eta_2 S}}}{\sqrt{m}}. \qquad (19)$$

As to the double signs in the above equations, we take a negative one to obtain a smaller upper bound. Correspondingly, a negative sign is taken also in double signs of Eqs. (15,16) hereafter.

The attainability conditions given by Eqs. (6a-6c) require that E_1, E_2, and E_3 take the following form:

$$E_\mu = \gamma_\mu \left(|\boldsymbol{\beta}_\mu| - \boldsymbol{\beta}_\mu \cdot \sigma\right), \ (\mu = 1, 2, 3),$$

where we defined

$$\boldsymbol{\beta}_1 \equiv \boldsymbol{\beta} - \frac{1}{2}\boldsymbol{a}_1, \ \boldsymbol{\beta}_2 \equiv \boldsymbol{\beta} - \frac{1}{2}\boldsymbol{a}_2, \ \boldsymbol{\beta}_3 \equiv \boldsymbol{\beta}.$$

The question is whether positive constants γ_1, γ_2, and γ_3 can be chosen so that the set $\{E_1, E_2, E_3\}$ respects the completeness condition of POVM given in Eq. (3b). This is possible if and only if a linear relation with positive coefficients exists for three vectors $\boldsymbol{\beta}_1$, $\boldsymbol{\beta}_2$, and $\boldsymbol{\beta}_3$.

$$c_1 \boldsymbol{\beta}_1 + c_2 \boldsymbol{\beta}_2 + c_3 \boldsymbol{\beta}_3 = 0, \ (c_1, c_2, c_3 \geq 0).$$

If such a linear relation exists, coefficients γ_μ can be constructed as $\gamma_\mu = \gamma c_\mu$ with an overall positive factor γ determined so that $\sum_\mu \gamma_\mu |\boldsymbol{\beta}_\mu| = 1$.

Since each of the three vectors is expressed by the two Bloch vectors $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$, a linear relation, which is unique up to an overall factor, is straightforwardly found, with coefficients given by

$$c_1 = \frac{y}{y+1}\left(\sqrt{m} - \frac{\sqrt{\eta_1 \eta_2 S} - \eta_1}{\sqrt{1 - 2\sqrt{\eta_1 \eta_2 S}}}\right),$$

$$c_2 = \frac{y}{y+1}\left(\sqrt{m} - \frac{\sqrt{\eta_1 \eta_2 S} - \eta_2}{\sqrt{1 - 2\sqrt{\eta_1 \eta_2 S}}}\right),$$

$$c_3 = \sqrt{\frac{\eta_1 \eta_2 S}{m}} - \sqrt{m} - \sqrt{1 - 2\sqrt{\eta_1 \eta_2 S}}.$$

Signs of c_μ vary depending on η_1, η_2, S, and m. Remember that we assumed $\eta_1 \leq \eta_2$. Then c_2 is always positive. We find that c_1 is positive if $m \geq m'_c$ and c_3 is positive if $m \leq m_c$, with m_c and m'_c defined in Eqs. (9) and (10), respectively. Thus, the set $\{E_1, E_2, E_3\}$ is a POVM if the error margin is in the range $m'_c \leq m \leq m_c$.

Remaining conditions are Eq. (3c) and Eq. (6d), which are reduced to $p_\times = m$ since $y \geq 1$. We can explicitly verify that the relation $p_\times = m$ holds after a long calculation by using the POVM constructed above. This is not a coincidence, but a consequence of how we determined parameter y. Parameter y was determined so that the upper bound d given by Eq. (17) is minimized:

$$\frac{\partial}{\partial y}d = \frac{\partial}{\partial y}\mathrm{tr}Y + m = 0.$$

We can show that $\frac{\partial}{\partial y}\mathrm{tr}Y = -p_\times$, which means that minimization of d leads to the relation $p_\times = m$. This can be seen in the following way. Suppose two positive semidefinite operators $A(y)$ and $B(y)$ depend on a variable y and satisfy $A(y)B(y) = 0$. Then we can show

$\mathrm{tr}A(y)B'(y) = 0$. To prove this, we define a function $f(x)$ to be

$$f(x) \equiv \mathrm{tr}A(y)B(y+x).$$

Note that $f(0) = 0$ while $f(x) \geq 0$ for any x, which implies that $f(x)$ has a minimum at $x = 0$. From $f'(0) = 0$, the desired result immediately follows. Now, operators Y_1, Y_2, Y and POVM elements E_μ are all positive semidefinite and satisfy $E_1Y_1 = E_2Y_2 = E_3Y = 0$. We therefore obtain

$$\begin{aligned}
\frac{\partial}{\partial y}\mathrm{tr}Y &= \mathrm{tr}E_1\frac{\partial Y}{\partial y} + \mathrm{tr}E_2\frac{\partial Y}{\partial y} + \mathrm{tr}E_3\frac{\partial Y}{\partial y} \\
&= \mathrm{tr}E_1\frac{\partial Y_1}{\partial y} + \mathrm{tr}E_2\frac{\partial Y_2}{\partial y} + \mathrm{tr}E_3\frac{\partial Y}{\partial y} - p_\times \\
&= -p_\times.
\end{aligned}$$

Thus, if error margin m is in the range $m'_c \leq m \leq m_c$, the upper bound of Eq. (18) is attained and the maximum success probability is given by

$$p_{\max} = \left(\sqrt{m} + \sqrt{1 - 2\sqrt{\eta_1\eta_2 S}}\right)^2. \qquad (20)$$

By using the optimal POVM, we find that the following symmetries turn out to hold:

$$P_{\rho_2|E_1} = P_{\rho_1|E_2}, \qquad (21)$$

$$P_{\rho_1|E_3} = P_{\rho_2|E_3}, \qquad (22)$$

where we introduced conditional probabilities defined by

$$P_{\rho_a|E_\mu} \equiv \frac{P_{\rho_a,E_\mu}}{P_{E_\mu}}, \;\; P_{E_\mu} \equiv P_{\rho_1,E_\mu} + P_{\rho_2,E_\mu}.$$

This is noteworthy, since in the problem, there is no apparent symmetry between ρ_1 and ρ_2 with general occurrence probabilities. The symmetry between two conditional error probabilities given by Eq. (21) will be important in Sec. V.

Before concluding the section, we present a simple argument to clarify how these symmetries emerge. Let us define two vectors $\boldsymbol{C}$ and $\boldsymbol{X}$ to be

$$\boldsymbol{C} \equiv \left(\sqrt{P_{\rho_1,E_1}}, \sqrt{P_{\rho_2,E_2}}\right),$$

$$\boldsymbol{X} \equiv \left(\sqrt{P_{\rho_2,E_1}}, \sqrt{P_{\rho_1,E_2}}\right).$$

The success probability $p_\circ$ is then given by $|\boldsymbol{C}|^2$. By the triangle inequality we observe

$$\sqrt{p_\circ} = |\boldsymbol{C}| \leq |\boldsymbol{X}| + |\boldsymbol{C} - \boldsymbol{X}|. \qquad (23)$$

Note $|\boldsymbol{X}| = \sqrt{p_\times}$, which must not exceed $\sqrt{m}$. An upper bound of $|\boldsymbol{C} - \boldsymbol{X}|$ can be determined in the following way.

$$\begin{aligned}
|\boldsymbol{C} - \boldsymbol{X}|^2 &= \boldsymbol{C}^2 + \boldsymbol{X}^2 - 2\boldsymbol{C}\cdot\boldsymbol{X} \\
&= 1 - P_{\rho_1,E_3} - P_{\rho_2,E_3} - 2\boldsymbol{C}\cdot\boldsymbol{X} \\
&\leq 1 - 2\left(\sqrt{P_{\rho_1,E_3}P_{\rho_2,E_3}} + \boldsymbol{C}\cdot\boldsymbol{X}\right) \\
&= 1 - 2\sqrt{\eta_1\eta_2}\sum_{\mu=1}^{3}\sqrt{q_\mu^{(1)}q_\mu^{(2)}},
\end{aligned}$$

where we used the inequality of arithmetic and geometric means

$$P_{\rho_1,E_3} + P_{\rho_2,E_3} \geq 2\sqrt{P_{\rho_1,E_3}P_{\rho_2,E_3}}, \qquad (24)$$

and we defined two probability distributions $q_\mu^{(1)}$ and $q_\mu^{(2)}$ by

$$q_\mu^{(a)} \equiv \mathrm{tr}\rho_a E_\mu \;\; (a = 1, 2, \;\; \mu = 1, 2, 3).$$

Expression $\sum_{\mu=1}^{3}\sqrt{q_\mu^{(1)}q_\mu^{(2)}}$ is the fidelity of two classical probability distributions $q_\mu^{(1)}$ and $q_\mu^{(2)}$ of obtaining measurement outcome μ for the two state ρ_1 and ρ_2. This classical fidelity is known to be lower-bounded by the quantum fidelity of the two states $|\langle\phi_1|\phi_2\rangle| = \sqrt{S}$ [17]. Thus, we obtain an upper bound for $p_\circ$ as

$$p_\circ \leq \left(\sqrt{m} + \sqrt{1 - 2\sqrt{\eta_1\eta_2 S}}\right)^2.$$

We notice that this is the attainable maximum given by Eq. (20). Consequently, equality must holds in all inequalities used to obtain this upper bound. Among them, equality of the triangle inequality in Eq. (23) implies vectors $\boldsymbol{C}$ and $\boldsymbol{X}$ are in the same direction, which immediately leads to the symmetry of Eq. (21). Equality of inequality (24) requires the relation of Eq. (22).

IV. SINGLE-STATE DOMAIN

In unambiguous discrimination ($m = 0$), omitting one of the states to be discriminated is optimal if its occurrence probability is sufficiently small. In discrimination with general error margin, a similar situation occurs in a domain of parameters (η_1 and m), which we call single-state domain. In this section, we will determine the optimal success probability in the single-state domain.

Assuming $\eta_1 \leq \eta_2$, we search for optimal POVM with $E_1 = 0$. Remember that all POVM elements are of rank 1 at most. We immediately see that E_2 and E_3 must constitute a projective measurement with respect to a set of orthonormal states $|\boldsymbol{f}\rangle$ and $|-\boldsymbol{f}\rangle$, with $\boldsymbol{f}$ being a unit Bloch vector to be determined.

$$E_2 = |\boldsymbol{f}\rangle\langle\boldsymbol{f}|, \qquad (25)$$

$$E_3 = |-\boldsymbol{f}\rangle\langle-\boldsymbol{f}|. \qquad (26)$$

Now look at the attainability conditions Eq. (6). Equation (6a) is trivially satisfied. Equations (6b) and (6c) require that

$$Y = \lambda_+ |\,\boldsymbol{f}\,\rangle\langle\,\boldsymbol{f}\,|, \tag{27}$$

$$Y - (\eta_2\rho_2 - y\eta_1\rho_1) = -\lambda_- |-\boldsymbol{f}\,\rangle\langle -\boldsymbol{f}\,|, \tag{28}$$

where λ_+ and λ_- are constants. We see that $\lambda_+ \geq 0$ and $\lambda_- \leq 0$ from upper bound conditions Eq. (4a) and Eq. (4c). Eliminating Y from Eqs. (27) and (28), we find

$$\eta_2\rho_2 - y\eta_1\rho_1 = \lambda_+ |\,\boldsymbol{f}\,\rangle\langle\,\boldsymbol{f}\,| + \lambda_- |-\boldsymbol{f}\,\rangle\langle -\boldsymbol{f}\,|,$$

which is the spectral decomposition of operator $\eta_2\rho_2 - y\eta_1\rho_1$. This shows that λ_+ and λ_- are the positive and negative eigenvalues with eigenstates $|\,\boldsymbol{f}\,\rangle$ and $|-\boldsymbol{f}\,\rangle$, respectively. We thus obtain λ_+, λ_-, and $\boldsymbol{f}$ in terms of Bloch vectors $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$.

$$\lambda_\pm = \frac{1}{2}(\eta_2 - y\eta_1) \pm \frac{1}{2}|\boldsymbol{a}_2|, \tag{29}$$

$$\boldsymbol{f} = \frac{\boldsymbol{a}_2}{|\boldsymbol{a}_2|}, \tag{30}$$

where $\boldsymbol{a}_2 = \eta_2\boldsymbol{n}_2 - y\eta_1\boldsymbol{n}_1$ as defined in Eq. (13).

Parameter y still remains to be determined. This can be done by requiring conditions Eqs. (3c), (4b), (4d), and (6d), which have not been checked so far.

The positivity of $Y - (\eta_1\rho_1 - y\eta_2\rho_2)$ of Eq. (4b) can be expressed as

$$\lambda_+ - (\eta_1 - y\eta_2) \geq \left|\lambda_+\frac{\boldsymbol{a}_2}{|\boldsymbol{a}_2|} - \boldsymbol{a}_1\right|,$$

where $\boldsymbol{a}_1 = \eta_1\boldsymbol{n}_1 - y\eta_2\boldsymbol{n}_2$ and $\boldsymbol{a}_2 = \eta_2\boldsymbol{n}_2 - y\eta_1\boldsymbol{n}_1$. After a rather involved calculation, we find that this condition together with positivity of y, Eq. (4d), imply the occurrence probabilities must satisfy an inequality given by

$$\eta_1 \leq \eta_2 S, \tag{31}$$

and parameter y an inequality given by

$$y \geq 1 + \frac{(1 - 2\sqrt{\eta_1\eta_2 S})(1 + \sqrt{\frac{\eta_2 S}{\eta_1}})}{\eta_2 S - \eta_1}. \tag{32}$$

The remaining conditions Eqs. (3c) and (6d) are simply reduced to a single equation $p_\times = m$ since $y \geq 1$ by Eq. (32). The average probability of error $p_\times$ is calculated as

$$p_\times = \eta_1 \mathrm{tr} E_2 \rho_1 = \eta_1 \frac{1 + \boldsymbol{f} \cdot \boldsymbol{n}_1}{2},$$

which should be equated to error margin m. This establishes a relation between parameter y and error margin m.

$$y = \frac{\eta_2}{\eta_1}\left(T - S + \sqrt{ST}\frac{\eta_1 - 2m}{\sqrt{m(\eta_1 - m)}}\right). \tag{33}$$

We can now translate the allowed range of parameter y given in Eq. (32) to that of error margin m. We find that the allowed range of error margin is given by

$$0 \leq m \leq \frac{(\eta_1 - \sqrt{\eta_1\eta_2 S})^2}{1 - 2\sqrt{\eta_1\eta_2 S}}. \tag{34}$$

Combining this with the condition (31), we see that the single-state domain is specified by inequality $0 \leq m \leq m'_c$, with m'_c defined in Eq. (10).

The optimal success probability in the single-state domain is obtained by calculating $d = \mathrm{tr} Y + ym$.

$$p_{\max} = \eta_2\left(\sqrt{\frac{m}{\eta_1}}S + \sqrt{\frac{\eta_1 - m}{\eta_1}}T\right)^2. \tag{35}$$

Note that, when $m = 0$, this reproduces the well-known result $p_\circ^{\max} = \eta_2(1 - S) = \eta_2(1 - |\langle\phi_1|\phi_2\rangle|^2)$ for unambiguous discrimination in the case of $\eta_1 \leq \eta_2 S$.

We assumed that $\eta_1 \leq \eta_2$. For the case of $\eta_1 \geq \eta_2$, it is clear that there is also a similar single-state domain, where E_2 is zero and state ρ_2 is omitted.

V. WEAK AND STRONG ERROR-MARGIN CONDITIONS

Until this point, we considered the discrimination problem with an error margin imposed on the average probability of error $p_\times$. We can consider a different way of imposing an error margin. Suppose the measurement outcome is $\mu = 1$. The probability of error in this case is the conditional probability $P_{\rho_2|E_1}$. In this section, we consider a discrimination problem with the conditions that the two conditional error probabilities must not exceed a certain error margin m.

$$P_{\rho_2|E_1} \leq m, \tag{36a}$$

$$P_{\rho_1|E_2} \leq m. \tag{36b}$$

These conditions are stronger than the error-margin condition, Eq. (3c), considered in preceding sections in the sense that Eq. (3c) follows from Eqs. (36).

$$p_\times = P_{\rho_2,E_1} + P_{\rho_1,E_2} = P_{\rho_2|E_1}P_{E_1} + P_{\rho_1|E_2}P_{E_2}$$
$$\leq m(P_{E_1} + P_{E_2}) \leq m.$$

We call the conditions given by Eqs. (36) and Eq. (3c) strong and weak error-margin conditions, respectively.

For equal occurrence probabilities, optimal solutions have already been obtained for both the weak and strong error-margin conditions [13]. In the following, we will establish a relation between optimal solutions of the two error-margin conditions for general occurrence probabilities.

In order to distinguish the two schemes, "strong" and "weak", we use superscripts S and W, respectively. Let us start with the optimal POVM $E_\mu^S(m^S)$ with strong

error-margin m^S. Suppose we calculate average error probability by using $E_\mu^\mathrm{S}(m^\mathrm{S})$, which we denote by $p_\times^\mathrm{S}(m^\mathrm{S})$. Using conditional error probabilities, we observe

$$
\begin{aligned}
p_\times^\mathrm{S}(m^\mathrm{S}) &= P_{\rho_2,E_1}^\mathrm{S}(m^\mathrm{S}) + P_{\rho_1,E_2}^\mathrm{S}(m^\mathrm{S}) \\
&= P_{\rho_2|E_1}^\mathrm{S}(m^\mathrm{S})P_{E_1}^\mathrm{S}(m^\mathrm{S}) + P_{\rho_1|E_2}^\mathrm{S}(m^\mathrm{S})P_{E_2}^\mathrm{S}(m^\mathrm{S}) \\
&\le m^\mathrm{S}\left(P_{E_1}^\mathrm{S}(m^\mathrm{S}) + P_{E_2}^\mathrm{S}(m^\mathrm{S})\right) \\
&= m^\mathrm{S}\left(p_\mathrm{max}^\mathrm{S}(m^\mathrm{S}) + p_\times^\mathrm{S}(m^\mathrm{S})\right),
\end{aligned}
$$

from which it follows that

$$
p_\times^\mathrm{S}(m^\mathrm{S}) \le \frac{m^\mathrm{S}}{1-m^\mathrm{S}}p_\mathrm{max}^\mathrm{S}(m^\mathrm{S}).
$$

This implies that the optimal POVM $E_\mu^\mathrm{S}(m^\mathrm{S})$ with strong error-margin m^S satisfies the weak error-margin condition with $m^\mathrm{W} = \frac{m^\mathrm{S}}{1-m^\mathrm{S}}p_\mathrm{max}^\mathrm{S}(m^\mathrm{S})$. Consequently, we obtain an inequality for two optimal success probabilities $p_\mathrm{max}^\mathrm{S}$ and $p_\mathrm{max}^\mathrm{W}$.

$$
p_\mathrm{max}^\mathrm{S}(m^\mathrm{S}) \le p_\mathrm{max}^\mathrm{W}\left(\frac{m^\mathrm{S}}{1-m^\mathrm{S}}p_\mathrm{max}^\mathrm{S}(m^\mathrm{S})\right). \tag{37}
$$

Note that the relation $p_\mathrm{max}^\mathrm{S}(m) \le p_\mathrm{max}^\mathrm{W}(m)$ holds for a common value of error margin m, because the strong error-margin conditions are stronger than the weak error-margin condition. Here, however, inequality (37) involves error margins of different values, and it will be shown that equality actually holds in this inequality.

We can derive another inequality for the two optimal success probabilities. Let us take the optimal POVM $E_\mu^\mathrm{W}(m^\mathrm{W})$ satisfying a weak error margin m^W. Remember that the two conditional probabilities of error are equal in the minimum-error and intermediate domains; $P_{\rho_2|E_1}^\mathrm{W} = P_{\rho_1|E_2}^\mathrm{W}$. In the single-state domain, one of the two conditional error probabilities is not defined. However, the following relations still hold with a constant κ:

$$
\begin{aligned}
P_{\rho_2,E_1}^\mathrm{W}(m^\mathrm{W}) &= \kappa P_{E_1}^\mathrm{W}(m^\mathrm{W}), \\
P_{\rho_1,E_2}^\mathrm{W}(m^\mathrm{W}) &= \kappa P_{E_2}^\mathrm{W}(m^\mathrm{W}).
\end{aligned}
$$

Adding these two expressions, we obtain

$$
\begin{aligned}
p_\times^\mathrm{W}(m^\mathrm{W}) &= \kappa\left(P_{E_1}^\mathrm{W}(m^\mathrm{W}) + P_{E_2}^\mathrm{W}(m^\mathrm{W})\right) \\
&= \kappa\left(p_\mathrm{max}^\mathrm{W}(m^\mathrm{W}) + p_\times^\mathrm{W}(m^\mathrm{W})\right),
\end{aligned}
$$

from which it follows that

$$
\begin{aligned}
\kappa &= \frac{p_\times^\mathrm{W}(m^\mathrm{W})}{p_\mathrm{max}^\mathrm{W}(m^\mathrm{W}) + p_\times^\mathrm{W}(m^\mathrm{W})} \\
&\le \frac{m^\mathrm{W}}{p_\mathrm{max}^\mathrm{W}(m^\mathrm{W}) + m^\mathrm{W}}.
\end{aligned}
$$

This shows that conditional error probabilities in the weak error-margin scheme satisfy the strong error margin conditions with $m^\mathrm{S} = \frac{m^\mathrm{W}}{p_\mathrm{max}^\mathrm{W}(m^\mathrm{W})+m^\mathrm{W}}$. We, therefore,

obtain another inequality given by

$$
p_\mathrm{max}^\mathrm{W}(m^\mathrm{W}) \le p_\mathrm{max}^\mathrm{S}\left(\frac{m^\mathrm{W}}{p_\mathrm{max}^\mathrm{W}(m^\mathrm{W}) + m^\mathrm{W}}\right). \tag{38}
$$

Actually equality holds in inequalities (37) and (38). This can be seen by their repeated uses as follows:

$$
\begin{aligned}
&p_\mathrm{max}^\mathrm{S}(m^\mathrm{S}) \\
&\le p_\mathrm{max}^\mathrm{W}\left(\frac{m^\mathrm{S}}{1-m^\mathrm{S}}p_\mathrm{max}^\mathrm{S}(m^\mathrm{S})\right) \\
&\le p_\mathrm{max}^\mathrm{S}\left(\frac{\frac{m^\mathrm{S}}{1-m^\mathrm{S}}p_\mathrm{max}^\mathrm{S}(m^\mathrm{S})}{p_\mathrm{max}^\mathrm{W}\left(\frac{m^\mathrm{S}}{1-m^\mathrm{S}}p_\mathrm{max}^\mathrm{S}(m^\mathrm{S})\right) + \frac{m^\mathrm{S}}{1-m^\mathrm{S}}p_\mathrm{max}^\mathrm{S}(m^\mathrm{S})}\right) \\
&\le p_\mathrm{max}^\mathrm{S}\left(\frac{\frac{m^\mathrm{S}}{1-m^\mathrm{S}}p_\mathrm{max}^\mathrm{S}(m^\mathrm{S})}{p_\mathrm{max}^\mathrm{S}(m^\mathrm{S}) + \frac{m^\mathrm{S}}{1-m^\mathrm{S}}p_\mathrm{max}^\mathrm{S}(m^\mathrm{S})}\right) \\
&= p_\mathrm{max}^\mathrm{S}(m^\mathrm{S}).
\end{aligned}
$$

In the above derivation, we used the fact that the success probability is an increasing function of error margin.

Thus, if two error margins m^S and m^W are related by

$$
m^\mathrm{S} = \frac{m^\mathrm{W}}{p_\mathrm{max}^\mathrm{W}(m^\mathrm{W}) + m^\mathrm{W}}, \tag{39}
$$

or equivalently by

$$
m^\mathrm{W} = \frac{m^\mathrm{S}}{1-m^\mathrm{S}}p_\mathrm{max}^\mathrm{S}(m^\mathrm{S}), \tag{40}
$$

the two optimal success probabilities are equal.

$$
p_\mathrm{max}^\mathrm{S}(m^\mathrm{S}) = p_\mathrm{max}^\mathrm{W}(m^\mathrm{W}). \tag{41}
$$

When one of the optimal success probabilities is known, the other can be determined by these equations. We note that the optimal POVMs are also related in the same way: $E_\mu^\mathrm{S}(m^\mathrm{S}) = E_\mu^\mathrm{W}(m^\mathrm{W})$.

Using the above relation, we obtain the optimal success probability with the strong error-margin conditions to be

$$
p_\mathrm{max}^\mathrm{S} =
\begin{cases}
\frac{1}{2}\left(1 + \sqrt{1 - 4\eta_1\eta_2 S}\right) & (m_c \le m \le 1), \\
A_m\left(1 - 2\sqrt{\eta_1\eta_2 S}\right) & (m_c' \le m \le m_c), \\
\frac{\eta_1\eta_2(1-m)(1-S)}{m\eta_2 + (1-m)\eta_1 - 2\sqrt{m(1-m)\eta_1\eta_2 S}} & (0 \le m \le m_c'),
\end{cases}
$$

where A_m is given by

$$
A_m = \frac{1-m}{(1-2m)^2}\left(1 + 2\sqrt{m(1-m)}\right).
$$

We assumed $\eta_1 \le \eta_2$, and m_c and m_c' are defined by

$$
m_c \equiv \frac{1}{2}\left(1 - \sqrt{1 - 4\eta_1\eta_2 S}\right),
$$

$$
m_c' \equiv
\begin{cases}
\dfrac{(\eta_1 - \sqrt{\eta_1\eta_2 S})^2}{(\eta_2 - \sqrt{\eta_1\eta_2 S})^2 + (\eta_1 - \sqrt{\eta_1\eta_2 S})^2} \\
\hspace{3cm} (\eta_1 \le \eta_2 S), \\
0 \quad (\eta_1 \ge \eta_2 S).
\end{cases}
$$

VI.   UPPER BOUND FOR MIXED STATE DISCRIMINATION WITH ERROR MARGIN

Let us consider that two states to be discriminated, ρ_1 and ρ_2, are mixed. The maximum success probability is known for minimum-error discrimination ($m = 1$). For unambiguous discrimination ($m = 0$) of general two mixed states, however, no analytic result for the maximum success probability is known. In Ref. [18], Rudolph et al. presented a simple upper bound for the success probability,

$$p_{\max}(\rho_1, \rho_2)$$
$$\leq \begin{cases} 1 - 2\sqrt{\eta_1\eta_2}F(\rho_1,\rho_2) & (\eta_1 \geq \eta_2 F(\rho_1,\rho_2)^2), \\ \eta_2\left(1 - F(\rho_1,\rho_2)^2\right) & (\eta_1 \leq \eta_2 F(\rho_1,\rho_2)^2). \end{cases} \quad (42)$$

where $F(\rho_1,\rho_2) = \mathrm{tr}(\sqrt{\rho_1}\rho_2\sqrt{\rho_1})^{1/2}$ is the fidelity of states ρ_1 and ρ_2. Later, the conditions for the two mixed states to reach the upper bound were analyzed and a new series of upper bounds was also found (see e.g., Refs. [19, 20, 21, 22]). For general error margin, a closed form of the maximum success probability is also hard to obtain as in unambiguous discrimination. However, it is likely that there exists an upper bound similar to Eq. (42), since it is expressed in terms of the fidelity of the two states and their occurrence probabilities. In the following, we will show that the method of Rudolph et al. can be applied to the case of general margin and an upper bound for success probability can easily be obtained by using the results of pure-state discrimination.

Suppose states to be discriminated are prepared in system Q, and purify the states by introducing another system R [17].

$$\rho_1^Q = \mathrm{tr}_R |\Psi_1^{QR}\rangle\langle\Psi_1^{QR}|, \quad (43)$$
$$\rho_2^Q = \mathrm{tr}_R |\Psi_2^{QR}\rangle\langle\Psi_2^{QR}|, \quad (44)$$

We assume that pure states $|\Psi_1^{QR}\rangle$ and $|\Psi_2^{QR}\rangle$ are chosen so that

$$|\langle\Psi_1^{QR}|\Psi_2^{QR}\rangle| = F(\rho_1,\rho_2),$$

which is always possible by Uhlmann's theorem [23].

Consider a hypothetical discrimination problem between pure states $|\Psi_1^{QR}\rangle$ and $|\Psi_2^{QR}\rangle$ with occurrence probability η_1 and η_2, respectively. We take the weak error-margin condition. The task is to maximize the success probability

$$p_\circ \equiv \eta_1 \mathrm{tr}_{QR} E_1^{QR} |\Psi_1^{QR}\rangle\langle\Psi_1^{QR}|$$
$$+ \eta_2 \mathrm{tr}_{QR} E_2^{QR} |\Psi_2^{QR}\rangle\langle\Psi_2^{QR}|, \quad (45)$$

under the condition that the average probability of error

$$p_\times \equiv \eta_1 \mathrm{tr}_{QR} E_2^{QR} |\Psi_1^{QR}\rangle\langle\Psi_1^{QR}|$$
$$+ \eta_2 \mathrm{tr}_{QR} E_1^{QR} |\Psi_2^{QR}\rangle\langle\Psi_2^{QR}|, \quad (46)$$

must not exceed error margin m. The maximum success probability for two pure states $|\phi_1\rangle$ and $|\phi_2\rangle$ is a function of $|\langle\phi_1|\phi_2\rangle|$ and independent of the dimension. We denote it by $p_{\max}^{\mathrm{pure}}(|\langle\phi_1|\phi_2\rangle|)$. The maximum success probability for the hypothetical discrimination problem is then given by $p_{\max}^{\mathrm{pure}}(F(\rho_1,\rho_2))$.

Let us impose an extra constraint on POVM E_μ^{QR} in this discrimination problem:

$$E_\mu^{QR} = E_\mu^Q \otimes \mathbf{1}^R, \ \mu = 1, 2, 3. \quad (47)$$

By this additional condition, the success probability (45) and the average error probability (46) are reduced to

$$p_\circ = \eta_1 \mathrm{tr}_Q E_1^Q \rho_1^Q + \eta_2 \mathrm{tr}_Q E_2^Q \rho_2^Q,$$
$$p_\times = \eta_1 \mathrm{tr}_Q E_2^Q \rho_1^Q + \eta_2 \mathrm{tr}_Q E_1^Q \rho_2^Q,$$

and the problem becomes equivalent to discrimination between the two mixed states ρ_1 and ρ_2 with occurrence probabilities η_1 and η_2. It is clear that any extra condition on POVM never increases the maximum success probability. Thus, we conclude that the success probability for two mixed states is upper-bounded by the maximum pure-state success probability with $|\langle\phi_1|\phi_2\rangle|$ replaced by the fidelity of the two mixed states.

$$p_{\max}(\rho_1,\rho_2) \leq p_{\max}^{\mathrm{pure}}(F(\rho_1,\rho_2)).$$

Using the results of pure-state discrimination given in Eq. (11), we obtain

$$p_{\max}(\rho_1,\rho_2)$$
$$\leq \begin{cases} \left(\sqrt{m} + \sqrt{1 - 2\sqrt{\eta_1\eta_2}F(\rho_1,\rho_2)}\right)^2 \\ \qquad\qquad (m_c' \leq m \leq m_c), \\ \eta_2\left(\sqrt{\frac{m}{\eta_1}}F(\rho_1,\rho_2) + \sqrt{\frac{\eta_1-m}{\eta_1}(1 - F(\rho_1,\rho_2)^2)}\right)^2 \\ \qquad\qquad (0 \leq m \leq m_c'), \end{cases}$$

where m_c and m_c' are given by

$$m_c = \frac{1}{2}\left(1 - \sqrt{1 - 4\eta_1\eta_2 F(\rho_1,\rho_2)^2}\right),$$
$$m_c' = \begin{cases} \dfrac{\left(\eta_1 - \sqrt{\eta_1\eta_2}F(\rho_1,\rho_2)\right)^2}{1 - 2\sqrt{\eta_1\eta_2}F(\rho_1,\rho_2)} & (\eta_1 \leq \eta_2 F(\rho_1,\rho_2)^2), \\ 0 & (\eta_1 \geq \eta_2 F(\rho_1,\rho_2)^2). \end{cases}$$

We assumed $\eta_1 \leq \eta_2$ as in the pure-state case.

For unambiguous discrimination ($m = 0$), the upper bound is reduced to the one given in Eq. (42). The maximum success probability of minimum-error discrimination is known and given by

$$p_{\max}(\rho_1,\rho_2) = \frac{1}{2}\left(1 + \mathrm{tr}|\eta_1\rho_1 - \eta_2\rho_2|\right),$$

which must not exceed our upper bound. This observation leads to an inequality

$$\mathrm{tr}|\eta_1\rho_1 - \eta_2\rho_2| \leq \sqrt{1 - 4\eta_1\eta_2 F(\rho_1,\rho_2)^2},$$

which is a generalization of the well-known inequality concerning the trace distance and the fidelity [17],

$$\frac{1}{2}\mathrm{tr}|\rho_1 - \rho_2| \leq \sqrt{1 - F(\rho_1, \rho_2)^2}.$$

VII. CONCLUDING REMARKS

In this paper, we considered a state discrimination problem which interpolates minimum-error and unambiguous discriminations by introducing a margin for the probability of error. In the case of two pure states with general occurrence probabilities, we obtained the optimal success probability in a fully analytic form.

Our final remark is about the possibility of optimal local discrimination between two multipartite pure states. Suppose two pure states are multipartite and generally entangled. An interesting question is whether the parties sharing the states can achieve the globally optimal success probability by local operations and classical communication (LOCC). It is known that two pure states can be optimally discriminated by LOCC in both the minimum-error [24, 25] and unambiguous [26, 27] discrimination schemes. For general error margin, we showed that this is also true when the occurrence probabilities are equal [13]. To show this, we proved the following general theorem [13]:

Theorem: *Let V be a two-dimensional subspace of a multipartite tensor-product space H, and P be the projector onto the subspace V. Then, for any three-element POVM $\{E_1, E_2, E_3\}$ of V with every element being of rank 0 or 1, there exists a one-way LOCC POVM $\{E_1^\mathrm{L}, E_2^\mathrm{L}, E_3^\mathrm{L}\}$ of H such that $E_\mu = PE_\mu^\mathrm{L}P$ ($\mu = 1, 2, 3$).*

This implies that a POVM satisfying the conditions of Theorem can be implemented by a one-way LOCC protocol as far as measurement for states in subspace V is concerned. As we have seen in Sec. II, for general occurrence probabilities, the optimal POVM elements are also of rank 1 at most. Thus, for any error margin and any occurrence probabilities, two multipartite pure states can be optimally discriminated by LOCC.

Acknowledgments

[1] A. Chefles, Contemp. Phys. **41**, 401 (2000).
[2] C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic Press, New York, 1976).
[3] I. D. Ivanovic, Phys. Lett. A **123**, 257 (1987).
[4] D. Dieks, Phys. Lett. A **126**, 303 (1988).
[5] A. Peres, Phys. Lett. A **128**, 19 (1988).
[6] G. Jaeger and A. Shimony, Phys. Lett. A **197**, 83 (1995).
[7] S. Croke, E. Andersson, S. M. Barnett, C. R. Gilson, and J. Jeffers, Phys. Rev. Lett. **96**, 070401 (2006).
[8] A. Chefles and S. M. Barnett, J. Mod. Opt. **45**, 1295 (1998).
[9] C.-W. Zhang, C-F. Li, and G-C. Guo, Phys. Lett. A **261**, 25 (1999).
[10] J. Fiurasek and M. Jezek, Phys. Rev. A **67**, 012321 (2003).
[11] Y. C. Eldar, Phys. Rev. A **67**, 042309 (2003).
[12] M. A. P. Touzel, R. B. A. Adamson, and A. M. Steinberg Phys. Rev. A **76**, 062314 (2007).
[13] A. Hayashi, T. Hashimoto, and M. Horibe, Phys. Rev. A **78**, 012333 (2008).
[14] L. Vandenberghe and S. Boyd, SIAM Rev. **38**, 49 (1996).
[15] Y. C. Eldar, A. Megretski, G. C. Verghese, IEEE Trans. Inform. Theory, 49, 1007 (2003).
[16] Yonina C. Eldar, IEEE Trans. Inform. Theory, 49, 446 (2003).
[17] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, England, 2000).
[18] T. Rudolph, R. W. Spekkens, and P. S. Turner, Phys. Rev. A **68**, 010301(R) (2003).
[19] Y. Feng, R. Duan, and M. Ying, Phys. Rev. A **70**, 012308 (2004).
[20] Ulrike Herzog and Janos A. Bergou, Phys. Rev. A **71**, 050301(R) (2005).
[21] P. Raynal and N. Lütkenhaus, Phys. Rev. A **72**, 022342 (2005).
[22] X.-F. Zhou, Y.-S. Zhang, G-C. Guo, Phys. Rev. A **75**, 052314 (2007).
[23] A. Uhlmann, Rep. Math. Phys. **9**, 273 (1976).
[24] J. Walgate, A. J. Short, L. Hardy, and V. Vedral, Phys. Rev. Lett. **85**, 4972 (2000).
[25] S. Virmani, M. F. Sacchi, M. B. Plenio, and D. Markham, Phys. Lett. A **288**, 62 (2001).
[26] Y.-X. Chen and D. Yang, Phys. Rev. A **65**, 022320 (2002).
[27] Z. Ji, H. Cao, and M. Ying, Phys. Rev. A **71**, 032323 (2005).